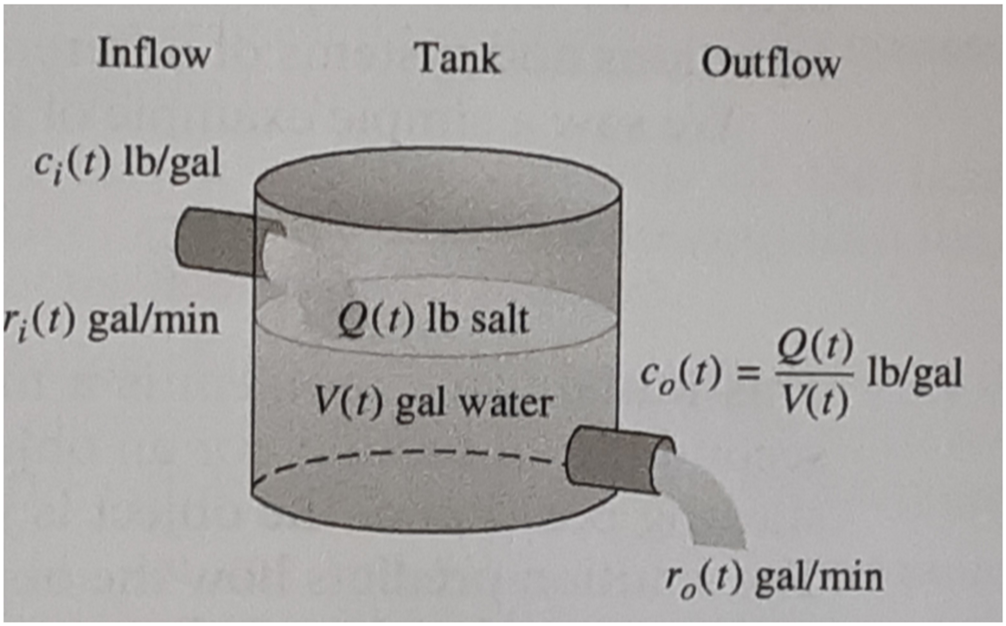
Mixing Problems Sect 2.3

Sec 2.3: Introduction to Math Modeling

Mixing Problems

We are looking for the amount of salt in the tank as function of time.



Where:

$Q(t) :=$ amount of salt (pounds) in the tank at time t (minutes).

$V(t) :=$ volume of water (gallons) in the tank at time t.

$c_i(t) :=$ inflow salt concentration at time t.

$c_o(t) :=$ outflow salt concentration at time t.

$r_i(t) :=$ inflow rate at time t.

$r_o(t) :=$ outflow rate at time t.

$\frac{Q(t)}{V(t)} = C_o$

$$Q'(t) = \underbrace{C_i r_i}_{\text{amount in}} - \underbrace{C_o r_o}_{\text{amount out}}$$

$V(0) = V_0$

$Q(0) = Q_0$

$$\text{Rate at which salt enters the tank} := r_i(t)c_i(t)$$
$$\text{Rate at which salt leaves the tank} := r_o(t)c_o(t)$$

where $c_o := \dfrac{Q(t)}{V(t)}$. Then, by conservation law:

$$\text{Rate of change of salt in the tank} := r_i(t)c_i(t) - r_o(t)c_o(t)$$

That is,

$$Q'(t) = r_i(t)c_i(t) - r_o(t)\frac{Q(t)}{V(t)}, \quad Q(0) = Q_0$$

$V'(t) = r_i - r_o$

$V(t) = \int_0^t (r_i(s) - r_o(s))\,ds + V(0)$

How to get $V(t)$? Notice that $V'(t) = r_i(t) - r_o(t)$. This yields,

$$V(t) - V(0) = \int_0^t r_i(s) - r_o(s)\ ds$$

$V(t) - V(0) = \int_0^t \underbrace{(r_i(s) - r_o(s))}_{\text{constant}}ds$

Having $V(t)$ we can solve

$$Q'(t) = r_i(t)c_i(t) - r_o(t)\frac{Q(t)}{V(t)}, \quad Q(0) = Q_0$$

$\Rightarrow V(t) = V(0) + (r_i - r_o)t$

$r_i = r_o \Rightarrow V(t) = V(0)$

Note: If $r_i(t) = r_o(t)$ for all t, then $V(t) = V(0)$ for all t.

constant volume

Ex.1 A tank initially contains 1000 gal of water in which is disolved 20 lb of salt. A valve is opened and water containing 0.2 lb of salt per galon flows into the tank at the rate of 5 gal/min. The mixture in the tank drains from the tank at the rate of 5 gal/min.
(a) Find $Q(t)$, the amount of salt in the tank after t minutes.

$Q'(t) = c_i r_i - c_o r_o = c_i r_i - \frac{Q(t) r_o}{V(t)}$

$V(t) = (r_i - r_o)t + V(0)$

$V(0) = 1000\,\text{gal}$ concentration in $\rightarrow c_i = 0.2\,\text{lb/gal}$

$Q(0) = 20\,\text{lb of salt}$ $r_i = 5\,\text{gal/min} = r_o$

$Q'(t) = 0.2 \cdot 5 - \frac{Q(t)}{V(t)} \cdot 5$

$V(t) = V(0) + (\underbrace{r_i - r_o}_{r_i = r_o = 5})t = V(0) = 1000\,\text{gal}$

$Q'(t) = 1 - \frac{\cancel{5}\,Q(t)}{\cancel{1000}\ 200} \Leftrightarrow Q' + \frac{Q}{200} = 1$

1st order, linear, non-homogeneous D.E

Solve using IF method

$Q' + \frac{Q}{200} = 1$ $\quad Q(0) = 20\,\text{lbs}$

① SF

② Int Factor $\mu(t) = e^{\int 1/200\,dt} = e^{t/200}$

③ Multi both side by $\mu(t)$

$(e^{t/200} Q(t))' = e^{t/200}$

④ Integrate both sides $e^{t/200} Q(t) = \int e^{t/200}\,dt$

$\Rightarrow e^{t/200} Q(t) = 200\,e^{t/200} + C$

⑤ Solve for $Q(t)$: $Q(t) = 200 + Ce^{-t/200}$

(b) Find the limiting value: $\lim_{t\to\infty} Q(t)$.

$\lim_{t\to\infty} 200 + C\underbrace{e^{-t/200}}_{\to 0}$

Limiting Amount: $\lim_{t\to\infty} = 200 = 200 + C \Rightarrow C = -180$

$Q(t) = 200 - 180e^{-t/200}$

Ex.2 Consider a 3000 gal tank that is 2/3 full of water. Also, assume that it has 100 lb of additive. If c_i =2 lb/gal, r_i =40gal/min and r_o=10gal/min, answer the following.

(a) What is the volume $V(t)$?

$V(t) = V(0) + \underbrace{(40 - 10)}_{r_i - r_o} t = 200 + 30t$

$Q(0) = 100$ lb of salt

$Q'(t) = \overbrace{c_i r_i}^{\text{in}} - \overbrace{c_o r_o}^{\text{out}} = 2 \times 40 - \frac{Q(t)}{V(t)} \times 10 = 80 - \frac{Q(t)}{200 + 30t} 10 \Rightarrow \boxed{Q'(t) + \frac{Q(t)}{200+3t} = 80}$

first order, linear, non-homogeneous

(b) Find $Q(t)$.

Solve by I.F.

① S.F.

② Build int factor $\mu(t) = e^{\int \frac{1}{200+3t} dt} = e^{\frac{1}{3}\ln(200+3t)} = e^{\ln(200+3t)^{1/3}} = (200+3t)^{1/3}$

$\boxed{m \log(a) = \log(a)^m}$

③ Mult both sides by $\mu(t)$

$Q(t) \cdot (200+3t)^{1/3} = 80(200+3t)^{1/3}$

④ Integrate both sides: $Q(t)(200+3t)^{1/3} = 80\int(200+3t)^{1/3} dt = \frac{80}{3}\int u^{1/3} du = \frac{80 \cdot 3}{3 \cdot 4} + C$ ← DNF

$Q(t)(200+3t)^{1/3} = 20(200+3t)^{4/3} + C$

$Q(t) = 20(200+3t) + C(200+3t)^{-1/3} \Rightarrow Q(0) = 100 = 20(200) + C(200^{-1/3})$

$u = 200 + 3t$

$du = 3dt$

$\frac{du}{3} = dt$

(c) When is the tank full?

when tank gets full, process stops $\underbrace{3000}_{\text{max}} = 2000 + 30t_{\text{max}} \Rightarrow \frac{1000}{30} = t_{\text{max}} \approx 30$ min